effectively scaled the eigenvalues of the preconditioning system to the same order of magnitude and results in the best numerical convergence. This is also true for a three-dimensional flow over a double-delta wing.[3]

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Application of a New K-τ Model to Near Wall Turbulent Flows

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Introduction

TWO-EQUATION turbulence models have become increasingly popular for the calculation of practical aerodynamic flows, which can have important applications in the design of advanced aircraft. The reason for this popularity is clear: two-equation models incorporate substantially more turbulence physics and require less ad hoc empiricisms than the older algebraic eddy-viscosity models without most of the added difficulties that arise from the computational implementation of second-order closure models. Among the existing two-equation models, the K-ϵ model is probably the most popular; its successes and limitations have been fairly well documented.[1,2] One major difficulty with the K-ϵ model that has yet to be fully resolved involves its application to near wall turbulent flows where inaccuracies and numerical stiffness problems can arise (see Ref. 3 for an interesting discussion of these problems). Recently, Speziale et al.[4] developed a new K-τ model for near wall turbulent flows wherein modeled transport equations are solved for the turbulent kinetic energy K and the turbulent time scale τ. Although the use of the turbulent time scale had been proposed earlier within the context of two-equation models (cf. Reynolds[5] and Zeierman and Wolfshtein[6]), none of these previous studies rendered a model that could be integrated directly to a solid boundary with the no-slip condition applied. Speziale et al.[4] demonstrated that their new K-τ model yields excellent results, and is computationally robust, when integrated directly to the wall in an incompressible flat-plate boundary layer with zero pressure gradient. The K-τ model alleviates the problem of the lack of natural boundary conditions for the dissipation rate in the K-ϵ model since the turbulent time scale τ vanishes identically at a solid boundary.

The purpose of the present Note is to provide a more comprehensive testing and evaluation of the K-τ model. Two test cases are chosen to evaluate the model: 1) the incompressible flat-plate boundary layer with adverse pressure gradients, and 2) incompressible flow past a backward facing step. Unlike in the previous study,[4] the K-τ model will be solved with an anisotropic as well as an isotropic eddy viscosity. Yoshizawa[7] and Speziale[8] have shown that, in some cases, the predictions of two-equation models can be enhanced considerably by the use of an anisotropic eddy viscosity. The results obtained from the K-τ model will be documented in detail and recommendations will be made for future research.

K-τ Model

In the standard form of the K-τ model, the Reynolds stress tensor $\tau_{ij} \equiv \overline{u_i' u_j'}$ (where u_i' is the fluctuating velocity) is of the form

$$\tau_{ij} = \frac{2}{3} K \delta_{ij} - \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{1}$$

where the eddy viscosity is given by

$$\nu_T = C_\mu f_\mu K \tau \tag{2}$$

and $\bar{u}_i$ is the mean velocity, $K \equiv \frac{1}{2} \overline{u_i' u_i'}$ the turbulent kinetic energy, $\epsilon = \nu \overline{\partial u_i' / \partial x_j \, \partial u_i' / \partial x_j}$ the turbulent dissipation rate, $\tau \equiv K/\epsilon$ the turbulent time scale, and C_μ a dimensionless con-

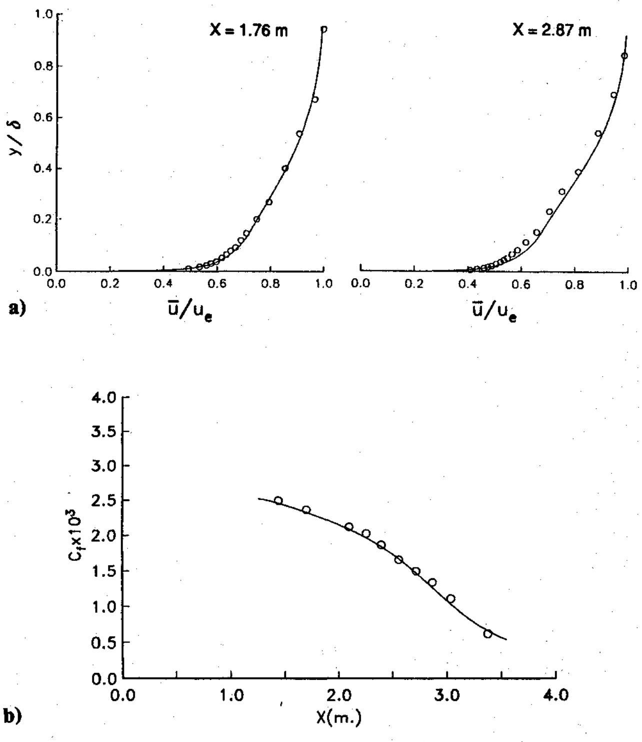

Fig. 1 Comparison of the results of the K-τ model and experiments for the flat-plate boundary layer with an adverse pressure gradient (—K-τ model; o experiments of Samuel and Joubert[9]): a) mean velocity; and b) skin friction coefficient.

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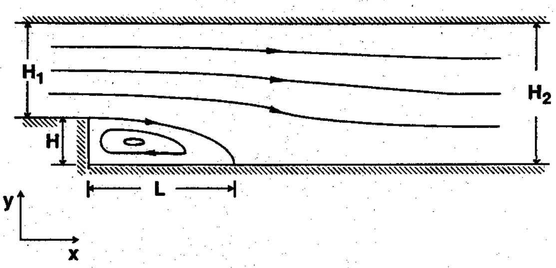

Fig. 2 Schematic diagram of turbulent flow past a backward-facing step.

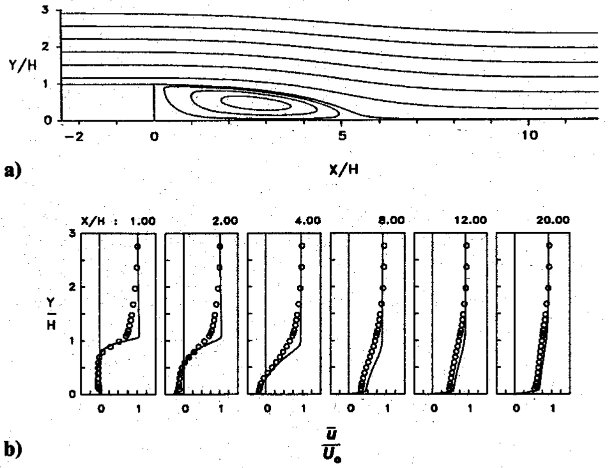

Fig. 3 Results obtained from the anisotropic K-τ model for incompressible turbulent flow past a backward-facing step with $H_1/H = 8$, $Re \approx 3.85 \times 10^4$ (—K-τ model; o experiments of Driver and Seegmiller[11]): a) mean velocity streamlines; b) mean velocity profiles.

stant that assumes a value of 0.09. In Eq. (2), f_μ is a wall damping function that is needed for asymptotic consistency[4]; it is taken to be of the form

$$f_\mu = [1 + (3.45)/(\sqrt{Re_t})] \tanh (y^+/70) \tag{3}$$

where $Re_t = K\tau/\nu$ is the turbulence Reynolds number and $y^+ = yu_\tau/\nu$ is the usual wall coordinate given that u_τ is shear velocity and ν is the kinematic viscosity. In an attached boundary layer, the shear velocity is usually taken to be $\sqrt{\tau_o/\rho}$ (where τ_o is the wall shear stress); for more complicated turbulent flows with separation, the shear velocity is assumed to be proportional to $K^{1/2}$. The wall damping function f_μ goes to unity sufficiently far from the wall. In the anisotropic version of the K-τ model, nonlinear corrections are added to the eddy viscosity model as developed by Speziale.[8] This leads to the following Reynolds stress model

$$\tau_{ij} = \tfrac{2}{3}K\delta_{ij} - 2C_\mu f_\mu K\tau\bar{S}_{ij} - 4C_D C_\mu^2 f_\mu^2 K\tau^2$$
$$\times (\bar{S}_{ik}\bar{S}_{kj} - \tfrac{1}{3}\bar{S}_{kl}\bar{S}_{kl}\delta_{ij} + \mathring{\bar{S}}_{ij} - \tfrac{1}{3}\mathring{\bar{S}}_{kk}\delta_{ij}) \tag{4}$$

where

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{5}$$

$$\mathring{\bar{S}}_{ij} = \frac{\partial \bar{S}_{ij}}{\partial t} + \bar{u}_k \frac{\partial \bar{S}_{ij}}{\partial x_k} - \frac{\partial \bar{u}_i}{\partial x_k}\bar{S}_{kj} - \frac{\partial \bar{u}_j}{\partial x_k}\bar{S}_{ki} \tag{6}$$

are, respectively, the mean rate of strain tensor and its frame-indifferent Oldroyd derivative, and C_D is a dimensionless constant that assumes a value of 1.68. Of course, the standard eddy-viscosity model (1) is recovered in the limit as $C_D \to 0$.

The Reynolds stress models (1) and (4) are solved in conjunction with modeled transport equations for K and τ, which take the form[4]:

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{K}{\tau} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_i}\right] \tag{7}$$

$$\frac{\partial \tau}{\partial t} + \bar{u}_i \frac{\partial \tau}{\partial x_i} = (C_{\epsilon 1} - 1)\frac{\tau}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} + (C_{\epsilon 2}f_2 - 1)$$
$$+ \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_{\tau 2}}\right)\frac{\partial \tau}{\partial x_i}\right] - \frac{2}{\tau}\left(\nu + \frac{\nu_T}{\sigma_{\tau 2}}\right)\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i}$$
$$+ \frac{2}{K}\left(\nu + \frac{\nu_T}{\sigma_{\tau 1}}\right)\frac{\partial K}{\partial x_i}\frac{\partial \tau}{\partial x_i} \tag{8}$$

where

$$f_2 = \left[1 - \frac{2}{9}\exp\left(-\frac{Re_t^2}{36}\right)\right]\left[1 - \exp\left(-\frac{y^+}{5}\right)\right]^2 \tag{9}$$

is a wall damping function that is needed for asymptotic consistency. The constants $C_{\epsilon 1}$ and $C_{\epsilon 2}$ assume the values of 1.44 and 1.83, respectively, whereas for their preliminary calculations, Speziale et al.[4] chose

$$\sigma_{\tau 1} = \sigma_{\tau 2} = \sigma_K = 1.36 \tag{10}$$

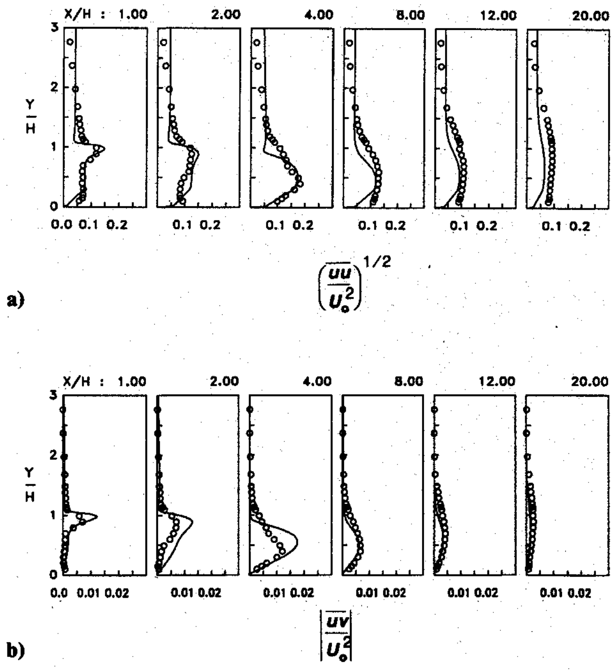

Fig. 4 Results obtained from the anisotropic K-τ model for incompressible turbulent flow past a backward-facing step with $H_1/H = 8$, $Re \approx 3.85 \times 10^4$ (—K-τ model, o experiments of Driver and Seegmiller[11]): a) turbulence intensity profiles; b) Reynolds shear stress profiles.

with the understanding that these constants could be fine tuned to a nonequivalent set of values when a future optimization over a variety of turbulent flows is conducted. When $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, and σ_K are identically equal, this K-τ model reduces to an equivalent K-ϵ model which can be simply obtained by replacing τ with K/ϵ in Eq. (7) and replacing the τ-transport equation (8) with the equivalent equation

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_i \frac{\partial \epsilon}{\partial x_i} = -C_{\epsilon 1} \frac{\epsilon}{K} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - C_{\epsilon 2} f_2 \frac{\epsilon^2}{K} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_i}\right] \tag{11}$$

where σ_ϵ is equal to the common value of $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, and σ_K given in Eq. (10).

Discussion of Results

The first problem that we will consider is the incompressible flat-plate boundary layer with adverse pressure gradients—the Samuel and Joubert[9] test case. This represents a rather severe test since the adverse pressure gradient is strong enough so that the flow is not that far removed from separation. In Fig. 1a, the mean velocity profiles predicted by the standard K-τ model using the eddy-viscosity representation (1) are compared with the experimental data[9] at two stations: $x = 1.76$ and 2.87 m (this corresponds to freestream Reynolds numbers Re_x in the range of 2×10^6–4×10^6). The model predictions agree to within 1% of the experimental results. In Fig. 1b, the skin friction predicted by the K-τ model is compared with the experimental data. Again, the agreement between the computations and the experimental data is within 1%. Unlike most other two-equation models, the K-τ model does not overpredict the skin friction for this adverse pressure gradient test case (see Ref. 10 for an interesting discussion of this issue). The use of an anisotropic eddy viscosity does not yield a significant change in the results for this case.

The second problem to be considered is incompressible turbulent flow past a backward-facing step—the same test case considered by Driver and Seegmiller[11] in their recent experiments. For this flow configuration, the aspect ratio H_1/H is 8 and the Reynolds number based on the inlet freestream velocity and step height is approximately 3.85×10^4 (see Fig. 2). The calculations are conducted on a 200×100 nonuniform mesh using a finite volume method (see Ref. 12). Here, we use the anisotropic eddy-viscosity model (4) since it has been recently demonstrated that such models give rise to improved predictions in recirculating flows.[8,12] In Fig. 3a, the computed streamlines obtained from the anisotropic K-τ model are shown. They indicate reattachment at approximately 6 step heights downstream of separation—a result that is within 5% of the experimental data.[11] In Fig. 3b, the mean velocity profiles obtained from the anisotropic K-τ model are compared with the experimental results; the computed results are in agreement to within 5% of the experimental results over 90% of the flowfield with a maximum deviation of about 10%. In Fig. 4a, the profiles of the turbulence intensity obtained from the anisotropic K-τ model are compared with the experimental data. Although there are some discrepancies between these results in the separation and recovery zone, on balance, the agreement is within 10% over 90% of the flowfield (a minimum discrepancy of 25% occurs in the separation zone). In Fig. 4b, the profiles of the Reynolds shear stress obtained from the anisotropic K-τ model are compared with the experimental data.[11] Again, although there are comparable discrepancies in the results within the separation and recovery zones, the model predictions agree to within 10% of the experimental data over most of the flowfield. There is no question that this near wall model does a substantially better job than the standard K-ϵ model, which underpredicts the reattachment point by an amount of the order of 25–30%.

Conclusions

The near wall K-τ model of Speziale et al.[4] has been tested in two applications involving incompressible turbulent flows, and the results obtained are quite encouraging. The standard K-τ model, with an isotropic eddy viscosity, yielded excellent results for the challenging incompressible adverse pressure gradient case of Samuel-Joubert.[9] For the separated flow considered, namely, turbulent flow past a backward-facing step, excellent results were obtained from the K-τ model based on an anisotropic eddy viscosity. In fact, we believe that these may be the most accurate calculations for this backstep problem that have yet been obtained from a two-equation model without the ad hoc adjustment of constants.

Future calculations are planned for supersonic boundary layers with streamline curvature and separation (shock induced or otherwise). For these flows, the K-τ model will most probably have to be applied with the anisotropic eddy-viscosity model (4). Furthermore, it may become necessary to incorporate dilatational effects into the model. After these calculations are completed and refinements are made accordingly, we will have a much better idea of the range of applicability of the K-τ model. Nonetheless, at this point, the prospects appear to be quite promising.

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